A decision procedure for $\mathcal{SHOIQ}$ with transitive closure of roles

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Abstract. The Semantic Web makes an extensive use of the OWL DL ontology language, underlied by the $\mathcal{SHOIQ}$ description logic, to formalize its resources. In this paper, we propose a decision procedure for this logic extended with the transitive closure of roles in concept axioms, a feature needed in several application domains. The most challenging issue we have to deal with when designing such a decision procedure is to represent infinitely non-tree-shaped models, which are different from those of $\mathcal{SHOIQ}$ ontologies. To address this issue, we introduce a new blocking condition for characterizing models which may have an infinite non-tree-shaped part.

1 Introduction

The ontology language OWL-DL [1] is widely used to formalize data resources on the Semantic Web. This language is mainly based on the description logic $\mathcal{SHOIN}$ which is known to be decidable [2]. Although $\mathcal{SHOIN}$ provides *transitive roles* to model transitivity of relations, we can find several applications in which the *transitive closure of roles*, that is more expressive than transitive roles, is needed. For instance, we consider an ontology, namely $\mathcal{O}_1$, that consists of the following axioms:

$\mathsf{Human} \sqsubseteq \exists\mathsf{hasAncestor}.\{\mathsf{Eva}\}$, where $\mathsf{hasAncestor}$ is transitive
$\mathsf{hasParent} \sqsubseteq \mathsf{hasAncestor}$, $\{\mathsf{Mike}\} \sqsubseteq \mathsf{Human}$, $\{\mathsf{Mike}\} \sqsubseteq \forall\mathsf{hasParent}.\bot$

We can see that $\mathcal{O}_1$ is consistent. However, the last axiom in $\mathcal{O}_1$ would be considered as a design error which should lead to inconsistency. If the transitive role "hasAncestor" is replaced with the transitive closure "$\mathsf{hasParent}^+$" (and the second axiom is removed), the first axiom becomes:

$$\mathsf{Human} \sqsubseteq \exists\mathsf{hasParent}^+.\{\mathsf{Eva}\}$$

It follows that the modified ontology is consistent. The point is that an instance of "$\mathsf{hasParent}^+$" represents exactly a sequence of instances of "hasParent" while an instance of "hasAncestor" corresponds to a sequence of instances of *itself*. In this paper, we consider an extension of $\mathcal{SHOIQ}$ by enabling transitive closure of roles in concept axioms. In the general case, transitive closure is not expressible in the first order logic [3], the logic from which DL is a sublanguage, while the second order logic is sufficiently expressive to do so.

In the DL literature ([4]; [5]), there have been works dealing with transitive closure of roles. Recently, Ortiz [5] has proposed an algorithm for deciding consistency in the logic $\mathcal{ALCQI}b^{+}_{reg}$ which allows for transitive closure of roles. However, nominals are disallowed in this logic. It is known that reasoning with a DL including number restrictions, inverse roles, nominals and transitive closure of roles is hard. The reason for this is that there exists an ontology in that DL whose models have an *infinite* non-tree-shaped part. Calvanese *et al.* [6] have presented an automata-based technique for dealing with the logic $\mathcal{ZOIQ}$ that includes transitive closure of roles, and showed that the sublogics $\mathcal{ZIQ}$, $\mathcal{ZOQ}$ and $\mathcal{ZOI}$ are decidable. To obtain this result, the authors have introduced the *quasi-forest model property* to characterize models of ontologies in these sublogics. Although they are very expressive, none of these sublogics includes $\mathcal{SHOIQ}$ with transitive closure of roles, namely $\mathcal{SHOIQ}_{(+)}$. The following example[3], noted $\mathcal{K}_1$, shows that there is an ontology in $\mathcal{SHOIQ}_{(+)}$ which does not enjoy the quasi-forest model property. We consider the following axioms:

(1) $\{o\} \sqsubseteq \mathsf{A}$; $\mathsf{A} \sqcap \mathsf{B} \sqsubseteq \bot$; $\mathsf{A} \sqsubseteq \exists \mathsf{R}.\mathsf{A} \sqcap \exists \mathsf{R}'.\mathsf{B}$; $\mathsf{B} \sqsubseteq \exists \mathsf{S}^{+}.\{o\}$

(2) $\{o\} \sqsubseteq \forall \mathsf{X}^{-}.\bot$; $\top \sqsubseteq\ \leq 1\,\mathsf{X}.\top$; $\top \sqsubseteq\ \leq 1\,\mathsf{X}^{-}.\top$ where $\mathsf{X} \in \{\mathsf{R}, \mathsf{R}', \mathsf{S}\}$

Figure 1 shows an infinite non-tree-shaped model of $\mathcal{K}_1$. In fact, each individual x that satisfies $\exists \mathsf{S}^{+}.\{o\}$ must have two distinct paths from x to the individual satisfying nominal o. Intuitively, we can see that (i) such a x must satisfy $\exists \mathsf{S}^{+}.\{o\}$ and B, (ii) an individual satisfying B must connect to another individual satisfying A which must have a R-path to nominal o, and (iii) two concepts A and B are disjoint.

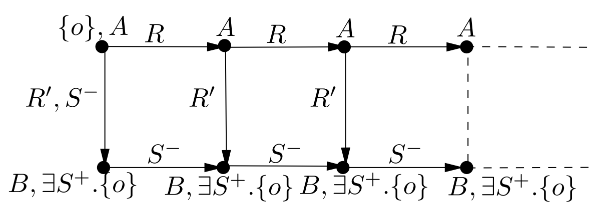

Fig. 1. An infinite non tree-shaped model of $\mathcal{K}_1$

This example shows that methods ([7], [8], [6]) based on the hypothesis which says that if an ontology is consistent it has a *quasi-forest model*, could fail to address the problem of consistency in a DL including simultaneously $\mathcal{O}$ (nominals), $\mathcal{I}$ (inverse roles), $\mathcal{Q}$ (number restrictions) and transitive closure of roles.

In this paper, we propose a decision procedure for the problem of consistency in $\mathcal{SHOIQ}$ with transitive closure of roles in concept axioms. The underlying idea of our algorithm is founded on the *star-type* and *frame* notions introduced by Pratt-Hartmann [9]. This technique uses star-types to represent individuals and "tiles" them together to form a frame for representing a model. For each star-type σ, we maintain a function $\delta(\sigma)$ which stores the number of individuals satisfying this star-type. To obtain termination, we introduce two additional structures for establishing a new blocking condition:

[3] This example is initially proposed by Sebastian Rudolph from an informal discussion

(i) the first one, namely *cycles*, describes duplicate parts of a model resulting from interactions of logic constructors in $\mathcal{SHOIQ}$, (ii) the second one, namely *blocking-blocked cycles*, describes parts of a model bordered by cycles which allow a frame to satisfy transitive closure of roles occurring in concepts of the form $\exists R^+.C$.

2 The Description Logic $\mathcal{SHOIQ}_{(+)}$

In this section, we present the syntax, the semantics and main inference problems of $\mathcal{SHOIQ}_{(+)}$. In addition, we introduce a tableau structure for $\mathcal{SHOIQ}_{(+)}$, which allows us to represent a model of a $\mathcal{SHOIQ}_{(+)}$ knowledge base.

Definition 1. *Let $\mathbf{R}$ be a non-empty set of* role names *and $\mathbf{R}_+ \subseteq \mathbf{R}$ be a set of transitive role names. We use $\mathbf{R}_\mathsf{I} = \{P^- \mid P \in \mathbf{R}\}$ to denote a set of inverse roles, and $\mathbf{R}_\oplus = \{Q^+ \mid Q \in \mathbf{R} \cup \mathbf{R}_\mathsf{I}\}$ to denote a set of transitive closure of roles. Each element of $\mathbf{R} \cup \mathbf{R}_\mathsf{I} \cup \mathbf{R}_\oplus$ is called a $\mathcal{SHOIQ}_{(+)}$-role. A* role inclusion axiom *is of the form $R \sqsubseteq S$ for two $\mathcal{SHOIQ}_{(+)}$-roles R and S such that $R \notin \mathbf{R}_\oplus$ and $S \notin \mathbf{R}_\oplus$. A* role hierarchy *$\mathcal{R}$ is a finite set of role inclusion axioms. An interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consists of a non-empty set $\Delta^\mathcal{I}$ (*domain*) and a function $\cdot^\mathcal{I}$ which maps each role name to a subset of $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$ such that*

$$R^{-\mathcal{I}} = \{\langle x,y\rangle \in \Delta^\mathcal{I} \times \Delta^\mathcal{I} \mid \langle y,x\rangle \in R^\mathcal{I}\} \textit{ for all } R \in \mathbf{R},$$
$$\langle x,z\rangle \in S^\mathcal{I}, \langle z,y\rangle \in S^\mathcal{I} \textit{ implies } \langle x,y\rangle \in S^\mathcal{I} \textit{ for each } S \in \mathbf{R}_+, \textit{ and}$$
$$(Q^+)^\mathcal{I} = \bigcup_{n>0} (Q^n)^\mathcal{I} \textit{ with } (Q^1)^\mathcal{I} = Q^\mathcal{I},$$
$$(Q^n)^\mathcal{I} = \{\langle x,y\rangle \in (\Delta^\mathcal{I})^2 \mid \exists z \in \Delta^\mathcal{I}, \langle x,z\rangle \in (Q^{n-1})^\mathcal{I}, \langle z,y\rangle \in Q^\mathcal{I}\} \textit{ for } Q^+ \in \mathbf{R}_\oplus$$

$*$ *An interpretation $\mathcal{I}$ satisfies a role hierarchy $\mathcal{R}$ if $R^\mathcal{I} \subseteq S^\mathcal{I}$ for each $R \sqsubseteq S \in \mathcal{R}$. Such an interpretation is called a* model *of $\mathcal{R}$, denoted by $\mathcal{I} \models \mathcal{R}$. To simplify notations for nested inverse roles and transitive closures of roles, we define two functions $\cdot^\ominus$ and $\cdot^\oplus$ as follows:*

$$R^\ominus = \begin{cases} R^- & \textit{if } R \in \mathbf{R}; \\ S & \textit{if } R = S^- \textit{ and } S \in \mathbf{R}; \\ (S^-)^+ & \textit{if } R = S^+,\ S \in \mathbf{R}, \\ S^+ & \textit{if } R = (S^-)^+,\ S \in \mathbf{R} \end{cases} \quad R^\oplus = \begin{cases} R^+ & \textit{if } R \in \mathbf{R}; \\ S^+ & \textit{if } R = (S^+)^+ \textit{ and } S \in \mathbf{R}; \\ (S^-)^+ & \textit{if } R = S^- \textit{ and } S \in \mathbf{R}; \\ (S^-)^+ & \textit{if } R = (S^+)^- \textit{ and } S \in \mathbf{R} \end{cases}$$

$*$ *A relation $\underline{\circledast}$ is defined as the transitive-reflexive closure $\mathcal{R}^+$ of $\sqsubseteq$ on $\mathcal{R} \cup \{R^\ominus \sqsubseteq S^\ominus \mid R \sqsubseteq S \in \mathcal{R}\} \cup \{R^\oplus \sqsubseteq S^\oplus \mid R \sqsubseteq S \in \mathcal{R}\} \cup \{Q \sqsubseteq Q^\oplus \mid Q \in \mathbf{R} \cup \mathbf{R}_\mathsf{I}\}$. We define a function* $\mathsf{Trans}(R)$ *which returns* true *iff there is some $Q \in \mathbf{R}_+ \cup \{P^\ominus \mid P \in \mathbf{R}_+\} \cup \{P^\oplus \mid P \in \mathbf{R} \cup \mathbf{R}_\mathsf{I}\}$ such that $Q \underline{\circledast} R \in \mathcal{R}^+$. A role R is called* simple *w.r.t. $\mathcal{R}$ if* $\mathsf{Trans}(R) = \mathsf{false}$.

The reason for the introduction of two functions $\cdot^\ominus$ and $\cdot^\oplus$ in Definition 1 is that they avoid using R^{--} and R^{++}. Moreover, it remains a unique nested case $(R^-)^+$. According to Definition 1, axiom $R \sqsubseteq Q^\oplus$ is not allowed in a role hierarchy $\mathcal{R}$ since this may lead to undecidability [10] even if R is simple. Notice that the closure $\mathcal{R}^+$ may contain $R \sqsubseteq Q^\oplus$ if $R \sqsubseteq Q$ belongs to $\mathcal{R}$.

Definition 2 (terminology). *Let* $\mathbf{C}$ *be a non-empty set of* concept names *with a non-empty subset* $\mathbf{C}_o \subseteq \mathbf{C}$ *of nominals. The set of* $\mathcal{SHOIQ}_{(+)}$*-concepts is inductively defined as the smallest set containing all* C *in* $\mathbf{C}$, $\top$, $C \sqcap D$, $C \sqcup D$, $\neg C$, $\exists R.C$, $\forall R.C$, $(\leq n\, S.C)$ *and* $(\geq n\, S.C)$ *where* n *is a positive integer,* C *and* D *are* $\mathcal{SHOIQ}_{(+)}$*-concepts,* R *is an* $\mathcal{SHOIQ}_{(+)}$*-role and* S *is a simple role w.r.t. a role hierarchy. We denote* $\perp$ *for* $\neg\top$*. The interpretation function* $\cdot^{\mathcal{I}}$ *of an interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *maps each concept name to a subset of* $\Delta^{\mathcal{I}}$ *such that* $\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$, $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$, $(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$, $(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \backslash C^{\mathcal{I}}$, $|\{o^{\mathcal{I}}\}| = 1$ *for all* $o \in \mathbf{C}_o$, $(\exists R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}}, \langle x, y\rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$, $(\forall R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \forall y \in \Delta^{\mathcal{I}}, \langle x, y\rangle \in R^{\mathcal{I}} \Rightarrow y \in C^{\mathcal{I}}\}$, $(\geq n\, S.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid |\{y \in C^{\mathcal{I}} \mid \langle x, y\rangle \in S^{\mathcal{I}}| \geq n\}$, $(\leq n\, S.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid |\{y \in C^{\mathcal{I}} \mid \langle x, y\rangle \in S^{\mathcal{I}}| \leq n\}$ *where* $|S|$ *is denoted for the cardinality of a set* S*. An axiom* $C \sqsubseteq D$ *is called a general concept inclusion (GCI) where* C, D *are* $\mathcal{SHOIQ}_{(+)}$*-concepts (possibly complex), and a finite set of GCIs is called a terminology* $\mathcal{T}$*. An interpretation* $\mathcal{I}$ *satisfies a GCI* $C \sqsubseteq D$ *if* $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ *and* $\mathcal{I}$ *satisfies a terminology* $\mathcal{T}$ *if* $\mathcal{I}$ *satisfies each GCI in* $\mathcal{T}$*. Such an interpretation is called a* model *of* $\mathcal{T}$*, denoted by* $\mathcal{I} \models \mathcal{T}$*. A pair* $(\mathcal{T}, \mathcal{R})$ *is called a* $\mathcal{SHOIQ}_{(+)}$ *knowledge base where* $\mathcal{R}$ *is a* $\mathcal{SHOIQ}_{(+)}$ *role hierarchy and* $\mathcal{T}$ *is a* $\mathcal{SHOIQ}_{(+)}$ *terminology. A knowledge base* $(\mathcal{T}, \mathcal{R})$ *is said to be consistent if there is a model* $\mathcal{I}$ *of both* $\mathcal{T}$ *and* $\mathcal{R}$*, i.e.,* $\mathcal{I} \models \mathcal{T}$ *and* $\mathcal{I} \models \mathcal{R}$*. A concept* C *is called* satisfiable *w.r.t.* $(\mathcal{T}, \mathcal{R})$ *iff there is some interpretation* $\mathcal{I}$ *such that* $\mathcal{I} \models \mathcal{R}$, $\mathcal{I} \models \mathcal{T}$ *and* $C^{\mathcal{I}} \neq \emptyset$*. Such an interpretation is called a* model *of* C *w.r.t.* $(\mathcal{T}, \mathcal{R})$*. A concept* D subsumes *a concept* C *w.r.t.* $(\mathcal{T}, \mathcal{R})$*, denoted by* $C \sqsubseteq D$*, if* $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ *holds in each model* $\mathcal{I}$ *of* $(\mathcal{T}, \mathcal{R})$*.* $\triangleleft$

Since unsatisfiability, subsumption and consistency w.r.t. a $\mathcal{SHOIQ}_{(+)}$ knowledge base can be reduced to each other, it suffices to study knowledge base consistency. For the ease of construction, we assume all concepts to be in *negation normal form* (NNF), i.e., negation occurs only in front of concept names. Any $\mathcal{SHOIQ}_{(+)}$-concept can be transformed to an equivalent one in NNF by using DeMorgan's laws and some equivalences as presented in [11]. According to [12], $\mathsf{nnf}(C)$ can be computed in polynomial time in the size of C. For a concept C, we denote the nnf of C by $\mathsf{nnf}(C)$ and the nnf of $\neg C$ by $\dot{\neg} C$. Let D be a $\mathcal{SHOIQ}_{(+)}$-concept in NNF. We define $\mathsf{cl}(D)$ to be the smallest set that contains all sub-concepts of D including D. For a knowledge base $(\mathcal{T}, \mathcal{R})$, we reuse $\mathsf{cl}(\mathcal{T}, \mathcal{R})$ introduced by Horrocks *et al.* [7] to denote all sub-concepts occurring in the axioms of $(\mathcal{T}, \mathcal{R})$ as follows:

$$\mathsf{cl}(\mathcal{T}, \mathcal{R}) = \bigcup_{C \sqsubseteq D \in \mathcal{T}} \mathsf{cl}(\mathsf{nnf}(\neg C \sqcup D), \mathcal{R}) \ \textit{where}$$

$$\mathsf{cl}(E, \mathcal{R}) = \mathsf{cl}(E) \cup \{\dot{\neg} C \mid C \in \mathsf{cl}(E)\} \ \cup \tag{1}$$

$$\{\forall S.C \mid (\forall R.C \in \mathsf{cl}(E), S \underline{\boxed{*}} R) \textit{ or } (\dot{\neg}\forall R.C \in \mathsf{cl}(E), S \underline{\boxed{*}} R) \ \textit{where } S \textit{ occurs in } \mathcal{T} \textit{ or } \mathcal{R}\} \ \cup \tag{2}$$

$$\bigcup_{\exists Q^{\oplus}.C \textit{ occurs in } \mathcal{T}} \mathsf{cl}(\exists Q.C \sqcup \exists Q.\exists Q^{\oplus}.C) \tag{3}$$

Since (1) consists of sub-concepts from $\mathcal{T}$ and (2) is formed from concepts in (1) by replacing a role or a logic constructor with respective another role occurring in $\mathcal{R}$ or another logic constructor, both of these sets are bounded by $\mathcal{O}(|(\mathcal{T},\mathcal{R})|)$. Thus, $\mathsf{cl}(\mathcal{T},\mathcal{R})$ is bounded by $\mathcal{O}(|(\mathcal{T},\mathcal{R})|)$.

We have $\mathsf{cl}(\mathcal{T},\mathcal{R})$ is bounded by $\mathcal{O}(|(\mathcal{T},\mathcal{R})|)$ [7]. To translate *star-type* and *frame* structures presented by Pratt-Hartmann (2005) for $\mathcal{C}^2$ into those for $\mathcal{SHOIQ}$, we need to add new sets of concepts, denoted $\mathsf{cl}_1(\mathcal{T},\mathcal{R})$ and $\mathsf{cl}_2(\mathcal{T},\mathcal{R})$, to the signature of a $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T},\mathcal{R})$.

$$\begin{aligned}\mathsf{cl}_1(\mathcal{T},\mathcal{R}) = &\{\leq mS.C \mid \{(\leq nS.C), (\geq nS.C)\} \cap \mathsf{cl}(\mathcal{T},\mathcal{R}) \neq \emptyset, 1 \leq m \leq n\} \cup \\ &\{\geq mS.C \mid \{(\leq nS.C), (\geq nS.C)\} \cap \mathsf{cl}(\mathcal{T},\mathcal{R}) \neq \emptyset, 1 \leq m \leq n\}\end{aligned}$$

For a generating concept $(\geq nS.C)$ and a set $I \subseteq \{0, \cdots, \lceil log\, n+1\rceil\}$, we denote $\mathscr{C}^I_{(\geq nS.C)} = \bigsqcap_{i\in I} C^i_{(\geq nS.C)} \sqcap \bigsqcap_{j\notin I} \neg C^j_{(\geq nS.C)}$ where $C^i_{(\geq nS.C)}$ are new concept names for $0 \leq i \leq \lceil log\, n+1\rceil$. We define $\mathsf{cl}_2(\mathcal{T},\mathcal{R})$ as follows:

$$\begin{aligned}\mathsf{cl}_2(\mathcal{T},\mathcal{R}) = &\{C^i_{(\geq S.C)} \mid (\geq nS.C) \in \mathsf{cl}(\mathcal{T},\mathcal{R}) \cup \mathsf{cl}_1(\mathcal{T},\mathcal{R}), 0 \leq i \leq \lceil log\, n+1\rceil\} \cup \\ &\{\mathscr{C}^I_{(\geq nS.C)} \mid (\geq nS.C) \in \mathsf{cl}(\mathcal{T},\mathcal{R}) \cup \mathsf{cl}_1(\mathcal{T},\mathcal{R}), I \subseteq \{0,\cdots,\lceil log\, n+1\rceil\}\}\end{aligned}$$

Remark 1. If numbers are encoded in binary then the number of new concept names $C^i_{(\geq nS.D)}$ for $0 \leq i \leq \lceil log\, n+1\rceil$, is bounded by $\mathcal{O}(|(\mathcal{T},\mathcal{R})|)$ since n is bounded by $\mathcal{O}(2^{|(\mathcal{T},\mathcal{R})|})$. This implies that $|\mathsf{cl}_2(\mathcal{T},\mathcal{R})|$ is bounded by $\mathcal{O}(|(\mathcal{T},\mathcal{R})|)$. Note that two concepts $\mathscr{C}^I_{(\geq nS.C)}$ and $\mathscr{C}^J_{(\geq nS.C)}$ are disjoint for all $I, J \subseteq \{0, \cdots, \lceil log\, n+1\rceil\}$, $I \neq J$. The concepts $\mathscr{C}_{(\exists S.C)}$ and $\mathscr{C}^I_{(\geq nS.C)}$ will be used for building chromatic star-types. This notion will be clarified after introducing the frame structure (Definition 6).

Finally, we denote $\mathbf{CL}(\mathcal{T},\mathcal{R}) = \mathsf{cl}(\mathcal{T},\mathcal{R}) \cup \mathsf{cl}_1(\mathcal{T},\mathcal{R}) \cup \mathsf{cl}_2(\mathcal{T},\mathcal{R})$, and use $\mathbf{R}(\mathcal{T},\mathcal{R})$ to denote the set of all role names occurring in $\mathcal{T},\mathcal{R}$ with their inverse. The definition of $\mathbf{CL}(\mathcal{T},\mathcal{R})$ is inspired from the Fischer-Ladner closure that was introduced in [13]. The closure $\mathbf{CL}(\mathcal{T},\mathcal{R})$ contains not only sub-concepts syntactically obtained from $\mathcal{T}$ but also sub-concepts that are semantically derived from $\mathcal{T}$ w.r.t. $\mathcal{R}$. For instance, if $\forall S.C$ is a sub-concept from $\mathcal{T}$ and $R \boxtimes S \in \mathcal{R}$ then $\forall R.C \in \mathbf{CL}(\mathcal{T},\mathcal{R})$.

To describe a model of a $\mathcal{SHOIQ}_{(+)}$ knowledge base in a more intuitive way, we use a tableau structure that expresses semantic constraints resulting directly from the logic constructors in $\mathcal{SHOIQ}_{(+)}$.

Definition 3. *Let $(\mathcal{T},\mathcal{R})$ be an $\mathcal{SHOIQ}_{(+)}$ knowledge base. A tableau T for $(\mathcal{T},\mathcal{R})$ is defined to be a triplet $(\mathbf{S},\mathcal{L},\mathcal{E})$ such that $\mathbf{S}$ is a set of individuals, $\mathcal{L}$: $\mathbf{S} \to 2^{\mathbf{CL}(\mathcal{T},\mathcal{R})}$ and $\mathcal{E}$: $\mathbf{R}(\mathcal{T},\mathcal{R}) \to 2^{\mathbf{S}\times\mathbf{S}}$. For all $s,t \in \mathbf{S}$, $C, C_1, C_2 \in \mathbf{CL}(\mathcal{T},\mathcal{R})$, and $R, S, Q^\oplus \in \mathbf{R}(\mathcal{T},\mathcal{R})$, T satisfies the following properties:*

P1 *If $C_1 \sqsubseteq C_2 \in \mathcal{T}$ and $s \in \mathbf{S}$ then $\mathsf{nnf}(\neg C_1 \sqcup C_2) \in \mathcal{L}(s)$;*
P2 *If $C \in \mathcal{L}(s)$, then $\neg C \notin \mathcal{L}(s)$;*
P3 *If $C_1 \sqcap C_2 \in \mathcal{L}(s)$, then $C_1 \in \mathcal{L}(s)$ and $C_2 \in \mathcal{L}(s)$;*
P4 *If $C_1 \sqcup C_2 \in \mathcal{L}(s)$, then $C_1 \in \mathcal{L}(s)$ or $C_2 \in \mathcal{L}(s)$;*
P5 *If $\forall S.C \in \mathcal{L}(s)$ and $\langle s,t\rangle \in \mathcal{E}(S)$, then $C \in \mathcal{L}(t)$;*
P6 *If $\exists S.C \in \mathcal{L}(s)$ then there is some $t \in \mathbf{S}$ such that $\langle s,t\rangle \in \mathcal{E}(S)$ and $\{C, \mathscr{C}_{(\exists S.C)}\} \subseteq \mathcal{L}(t)$;*
P7 *If $\forall S.C \in \mathcal{L}(s)$ and $\langle s,t\rangle \in \mathcal{E}(R)$ for $R \boxtimes S$ and $\mathsf{Trans}(R)$ then $\forall R.C \in \mathcal{L}(t)$;*
P8 *If $\exists Q^\oplus.C \in \mathcal{L}(s)$ then $(\exists Q.C \sqcup \exists Q.\exists Q^\oplus.C) \in \mathcal{L}(s)$ and there are $s_1, \cdots, s_{n-1}$*

$\in \mathbf{S}$ *such that* $\exists Q.C \in \mathcal{L}(s_0) \cup \mathcal{L}(s_{n-1})$, $\langle s_i, s_{i+1}\rangle \in \mathcal{E}(Q)$ *with* $0 \leq i < n-1$, $s_0 = s$ *and* $\exists Q^{\oplus}.C \in \mathcal{L}(s_j)$ *for all* $0 \leq j < n-1$.

P9 $\langle s,t\rangle \in \mathcal{E}(R)$ *iff* $\langle t,s\rangle \in \mathcal{E}(R^{\ominus})$;

P10 *If* $\langle s,t\rangle \in \mathcal{E}(R)$, $R \circledast S$ *then* $\langle s,t\rangle \in \mathcal{E}(S)$;

P11 *If* $(\geq n\, S\, C) \in \mathcal{L}(s)$ *then there are* $t_1, \cdots, t_n \in \mathbf{S}$ *such that* $\{C, \mathscr{C}^{I_i}_{(\geq nS.C)}\} \subseteq \mathcal{L}(t_i)$ *and* $\langle s, t_i\rangle \in \mathcal{E}(S)$ *for all* $1 \leq i \leq n$, *and* $I_j, I_k \subseteq \{0, \cdots, \lceil log\, n+1 \rceil\}$, $I_j \neq I_k$ *for all* $1 \leq j < k \leq n$;

P12 *If* $(\leq n\, S\, C) \in \mathcal{L}(s)$ *then* $|S^T(s,C)| \leq n$;

P13 *If* $(\leq n\, S\, C) \in \mathcal{L}(s)$ *and* $\langle s,t\rangle \in \mathcal{E}(S)$ *then* $\{C, \neg C\} \cap \mathcal{L}(t) \neq \emptyset$ *where* $S^T(s,C) := \{t \in \mathbf{S} | \langle s,t\rangle \in \mathcal{E}(S) \wedge C \in \mathcal{L}(t)\}$;

P14 *If* $o \in \mathcal{L}(s) \cap \mathcal{L}(t)$ *for some* $o \in \mathbf{C}_o$ *then* $s = t$.

P15 *For each* $o \in \mathbf{C}_o$, *if* o *occurs in* $\mathcal{T}$ *then there is* $s \in \mathbf{S}$ *such that* $o \in \mathcal{L}(s)$.

Note that the property P8 is added to deal with transitive closure of roles. The following lemma establishes the equivalence between a model of an ontology and a tableau.

Lemma 1. *Let* $(\mathcal{T}, \mathcal{R})$ *be a* $\mathcal{SHOIQ}_{(+)}$ *knowledge base.* $(\mathcal{T}, \mathcal{R})$ *is consistent iff there is a tableau for* $(\mathcal{T}, \mathcal{R})$.

A proof of Lemma 1 can be found in [14].

3 A Decision Procedure For $\mathcal{SHOIQ}_{(+)}$

This section starts by translating *star-type* and *frame* structures presented by Pratt-Hartmann (2005) for $\mathcal{C}^2$ into those for $\mathcal{SHOIQ}_{(+)}$.

Definition 4 (star-type). *Let* $(\mathcal{T}, \mathcal{R})$ *be a* $\mathcal{SHOIQ}_{(+)}$ *knowledge base. A* star-type *is a pair* $\sigma = \langle \lambda(\sigma), \xi(\sigma)\rangle$, *where* $\lambda(\sigma) \in 2^{\mathbf{CL}(\mathcal{T},\mathcal{R})}$ *is called* core label, $\xi(\sigma) = (\langle r_1, l_1\rangle, \cdots, \langle r_d, l_d\rangle)$ *is a d-tuple over* $2^{\mathbf{R}(\mathcal{T},\mathcal{R})} \times 2^{\mathbf{CL}(\mathcal{T},\mathcal{R})}$. *A pair* $\langle r, l\rangle$ *is a* ray *of* σ *if* $\langle r, l\rangle = \langle r_i, l_i\rangle$ *for some* $1 \leq i \leq d$. *We use* $\langle r(\rho), l(\rho)\rangle$ *to denote a ray* $\rho = \langle r, l\rangle$ *where* $r(\rho) = r$ *and* $l(\rho) = l$.

- *A star-type* σ *is* nominal *if* $o \in \lambda(\sigma)$ *for some* $o \in \mathbf{C}_o$.
- *A star-type* σ *is* chromatic *if* $\rho \neq \rho'$ *implies* $l(\rho) \neq l(\rho')$ *for two rays* ρ, ρ' *of* σ. *When a star-type* σ *is chromatic,* $\xi(\sigma)$ *can be considered as a set of rays.*
- *Two star-types* σ, σ' *are* equivalent *if* $\lambda(\sigma) = \lambda(\sigma')$, *and there is a bijection* π *between* $\xi(\sigma)$ *and* $\xi(\sigma')$ *such that* $\pi(\rho) = \rho'$ *implies* $r(\rho') = r(\rho)$ *and* $l(\rho') = l(\rho)$.

We denote Σ *for the set of all star-types for* $(\mathcal{T}, \mathcal{R})$. $\triangleleft$

Note that for a chromatic star-type σ, $\xi(\sigma)$ can be considered as a set of rays since rays are distinct and not ordered. We can think of a star-type σ as the set of individuals x satisfying all concepts in $\lambda(\sigma)$, and each ray ρ of σ corresponds to a "neighbor" individual x_i of x such that $r(\rho)$ is the label of the link between x and x_i; and x_i satisfies all concepts in $l(\rho)$. In this case, we say that x *satisfies* σ.

Definition 5 (valid star-type). *Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}_{(+)}$ knowledge base. Let σ be a star-type for $(\mathcal{T}, \mathcal{R})$ where $\sigma = \langle \lambda(\sigma), \xi(\sigma) \rangle$. The star-type σ is* valid *if σ is chromatic and the following conditions are satisfied:*

1. *If $C \sqsubseteq D \in \mathcal{T}$ then $\mathsf{nnf}(\neg C \sqcup D) \in \lambda(\sigma)$;*
2. *$\{A, \neg A\} \not\subseteq \lambda$ for every concept name A where $\lambda = \lambda(\sigma)$ or $\lambda = l(\rho)$ for each $\rho \in \xi(\sigma)$;*
3. *If $C_1 \sqcap C_2 \in \lambda(\sigma)$ then $\{C_1, C_2\} \subseteq \lambda(\sigma)$;*
4. *If $C_1 \sqcup C_2 \in \lambda(\sigma)$ then $\{C_1, C_2\} \cap \lambda(\sigma) \neq \emptyset$;*
5. *If $\exists R.C \in \lambda(\sigma)$ then there is some ray $\rho \in \xi(\sigma)$ such that $C \in l(\rho)$ and $R \in r(\rho)$;*
6. *If $(\leq nS.C) \in \lambda(\sigma)$ and there is some ray $\rho \in \xi(\sigma)$ such that $S \in r(\rho)$ then $C \in l(\rho)$ or $\dot{\neg} C \in l(\rho)$;*
7. *If $(\leq nS.C) \in \lambda(\sigma)$ and there is some ray $\rho \in \xi(\sigma)$ such that $C \in l(\rho)$ and $S \in r(\rho)$ then there is some $1 \leq m \leq n$ such that $\{(\leq mS.C), (\geq mS.C)\} \subseteq \lambda(\sigma)$;*
8. *For each ray $\rho \in \xi(\sigma)$, if $R \in r(\rho)$ and $R \underline{\boxast} S$ then $S \in r(\rho)$;*
9. *If $\forall R.C \in \lambda(\sigma)$ and $R \in r(\rho)$ for some ray $\rho \in \xi(\sigma)$ then $C \in l(\rho)$;*
10. *If $\forall R.D \in \lambda(\sigma)$, $S \underline{\boxast} R$, $\mathsf{Trans}(S)$ and $R \in r(\rho)$ for some ray $\rho \in \xi(\sigma)$ then $\forall S.D \in l(\rho)$;*
11. *If $\exists Q^{\oplus}.C \in \lambda(\sigma)$ then $(\exists Q.C \sqcup \exists Q.\exists Q^{\oplus}.C) \in \lambda(\sigma)$;*
12. *If $(\geq nS.C) \in \lambda(\sigma)$ then there are n distinct rays $\rho_1, \cdots, \rho_n \in \xi(\sigma)$ such that $\{C, \mathscr{C}^{I_i}_{(\geq nS.C)}\} \subseteq l(\rho_i)$, $S \in r(\rho_i)$ for all $1 \leq i \leq n$; and $I_j, I_k \subseteq \{0, \cdots, log\, n + 1\}$, $I_j \neq I_k$ for all $1 \leq j < k \leq n$;*
13. *If $(\leq nS.C) \in \lambda(\sigma)$ and there do not exist $n+1$ rays $\rho_0, \cdots, \rho_n \in \xi(\sigma)$ such that $C \in l(\rho_i)$ and $S \in r(\rho_i)$ for all $0 \leq i \leq n$.* $\lhd$

Roughly speaking, a star-type σ is valid if each individual x satisfies *semantically* all concepts in $\lambda(\sigma)$. In fact, each condition in Definition 5 represents the semantics of a constructor in $\mathcal{SHOIQ}_{(+)}$ except for transitive closure of roles. From valid star-types, we can "tile" a model instead of using expansion rules for generating nodes as described in tableau algorithms. Before presenting how to "tile" a model from star-types, we need some notation that will be used in the remainder of the paper.

Notation 1 *We call $\mathcal{P} = \langle (\sigma_1, \rho_1, d_1), \cdots, (\sigma_k, \rho_k, d_k) \rangle$ a* sequence *where $\sigma_i \in \Sigma$, $\rho_i \in \xi(\sigma_i)$ and $d_i \in \mathbb{N}$ for $1 \leq i \leq k$.*

- *$\mathsf{tail}(\mathcal{P}) = (\sigma_k, \rho_k, d_k)$, $\mathsf{tail}_\sigma(\mathcal{P}) = \sigma_k$, $\mathsf{tail}_\rho(\mathcal{P}) = \rho_k$, $\mathsf{tail}_\delta(\mathcal{P}) = d_k$ and $|\mathcal{P}| = k$. We denote $\mathcal{L}(\mathcal{P}) = \lambda(\mathsf{tail}_\sigma(\mathcal{P}))$.*
- *$\mathsf{p}^i(\mathcal{P}) = (\sigma_i, \rho_i, d_i)$, $\mathsf{p}^i_\sigma(\mathcal{P}) = \sigma_i$, $\mathsf{p}^i_\rho(\mathcal{P}) = \rho_i$ and $\mathsf{p}^i_\delta(\mathcal{P}) = d_i$ for each $1 \leq i \leq k$.*
- *an operation $\mathsf{add}(\mathcal{P}, (\sigma, \rho, d))$ extends $\mathcal{P}$ to a new sequence with $\mathsf{add}(\mathcal{P}, (\sigma, \rho, d)) = \langle \mathcal{P}, (\sigma, \rho, d) \rangle$.*

Definition 6 (frame). *Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}_{(+)}$ knowledge base. A* frame *for $(\mathcal{T}, \mathcal{R})$ is a tuple $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$, where*

1. *$\mathcal{N}$ is a set of valid star-types such that σ is not equivalent to σ' for all $\sigma, \sigma' \in \mathcal{N}$;*
2. *$\mathcal{N}_o \subseteq \mathcal{N}$ is a set of nominal star-types;*

3. *Ω is a function that maps each pair (σ, ρ) with $\sigma \in \mathcal{N}$ and $\rho \in \xi(\sigma)$ to a sequence $\Omega(\sigma,\rho) = \langle(\sigma_1,\rho_1,d_1),\cdots,(\sigma_m,\rho_m,d_m)\rangle$ with $\sigma_i \in \mathcal{N}$, $\rho_i \in \xi(\sigma_i)$, $d_i \in \mathbb{N}$ for $1 \leq i \leq m$ such that for each σ_i with $1 \leq i \leq m$, it holds that $l(\rho) = \lambda(\sigma_i)$, $l(\rho_i) = \lambda(\sigma)$ and $r(\rho_i) = r^-(\rho)$ where $r^-(\rho) = \{R^\ominus \mid R \in r(\rho)\}$.*
4. *δ is a function $\delta : \mathcal{N} \to \mathbb{N}$. By abuse of notation, we also use δ to denote a function which maps each pair (σ, ρ) with $\sigma \in \mathcal{N}$ and $\rho \in \xi(\sigma)$ into a number in $\mathbb{N}$, i.e., $\delta(\sigma, \rho) \in \mathbb{N}$.* $\lhd$

Since a frame cannot contain two equivalent star-type (Condition 1 in Definition 6), the number of different star-types in a frame is bounded. The following lemma provides such a bound.

Lemma 2. *Let $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta\rangle$ be a frame for a $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$. The number of different star-types is bounded by $\mathcal{O}(2^{2^{|(\mathcal{T},\mathcal{R})|}})$.*

The lemma is a consequence of the following facts : (i) the number of different core labels of star-types is bounded by $\mathcal{O}(|(\mathcal{T},\mathcal{R})|)$, (ii) the number of different ray labels of star-types is bounded by $\mathcal{O}(2^{|(\mathcal{T},\mathcal{R})|})$, and (iii) the number of different rays of a star-type is bounded by $\mathcal{O}(2^{|(\mathcal{T},\mathcal{R})|})$ due to binary coding of numbers.

The frame structure, as introduced in Definition 6, allows us to compress individuals of a model into star-types. For each star-type σ and each ray $\rho \in \xi(\sigma)$, a list $\Omega(\sigma, \rho)$ of triples (σ_i, ρ_i, d_i) with $\rho_i \in \xi(\sigma_i)$ is maintained where σ_i is a "neighbor" star-type of σ via $\rho \in \xi(\sigma)$, and d_i indicates the d_i-th "layer" of rays of σ_i. We can think a layer of rays of σ_i as an individual that connects to its neighbor individuals via the rays of σ_i. The following definition presents how to connect such layers to form paths in a frame.

Definition 7 (path). *Let $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta\rangle$ be a frame for a $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$. A* path *is inductively defined as follows:*

1. *A sequence $\langle\emptyset, (\sigma, \rho, 1)\rangle$ is a path, namely* nominal path, *if $\sigma \in \mathcal{N}_o$ and $\rho \in \xi(\sigma)$;*
2. *A sequence $\langle\mathcal{P}, (\sigma, \rho, d)\rangle$ with $\mathcal{P} \neq \emptyset$ and $\mathsf{tail}(\mathcal{P}) = (\sigma_0, \rho_0, d_0)$, is a path if $(\sigma, \rho, d) = \mathsf{p}^{d_0}(\Omega(\sigma_0, \rho'))$ for each $\rho' \neq \rho_0$. In this case, we say that $\langle\mathcal{P}, (\sigma, \rho, d)\rangle$ is the ρ'-*neighbor *of $\mathcal{P}$, and two paths $\mathcal{P}$, $\langle\mathcal{P}, (\sigma, \rho, d)\rangle$ are neighbors.*
   Additionally, if $\langle\mathcal{P}, (\sigma, \rho, d)\rangle$ is a ρ'-neighbor of $\mathcal{P}$ and $Q \in r(\rho')$ then $\langle\mathcal{P}, (\sigma, \rho, d)\rangle$ is a Q-neighbor of $\mathcal{P}$. In this case, we say that $\langle\mathcal{P}, (\sigma, \rho, d)\rangle$ is a Q-neighbor of $\mathcal{P}$, or $\mathcal{P}$ is a $Q^\ominus$-neighbor of $\langle\mathcal{P}, (\sigma, \rho, d)\rangle$.

We define $\mathcal{P} \sim \mathcal{P}'$ if $\mathsf{tail}_\sigma(\mathcal{P}) = \mathsf{tail}_\sigma(\mathcal{P}')$ and $\mathsf{tail}_\delta(\mathcal{P}) = \mathsf{tail}_\delta(\mathcal{P}')$. Since $\sim$ is an equivalence relation over the set of all paths, we use $\mathscr{P}$ to denote the set of all equivalence classes $[\mathcal{P}]$ of paths in $\mathcal{F}$. For $[\mathcal{P}], [\mathcal{Q}] \in \mathscr{P}$, we define:

1. *$[\mathcal{P}]$ is a neighbor (ρ'-neighbor) of $[\mathcal{Q}]$ if there are $\mathcal{P}' \in [\mathcal{P}]$ and $\mathcal{Q}' \in [\mathcal{Q}]$ such that $\mathcal{Q}'$ is a neighbor (ρ'-neighbor) of $\mathcal{P}'$;*
2. *$[\mathcal{Q}]$ is a reachable path of $[\mathcal{P}]$ via a ray $\rho \in \xi(\mathsf{tail}_\sigma(\mathcal{P}))$ if there are $[\mathcal{P}_1], \cdots, [\mathcal{P}_n] \in \mathscr{P}$ such that $[\mathcal{P}_i] \neq [\mathcal{P}_j]$ for $1 \leq i < j \leq n$, $[\mathcal{P}] = [\mathcal{P}_1]$, $[\mathcal{Q}] = [\mathcal{P}_n]$, $[\mathcal{P}_2]$ is the ρ-neighbor of $[\mathcal{P}_1]$, $[\mathcal{P}_{i+1}]$ is a neighbor of $[\mathcal{P}_i]$ for all $1 \leq i < n-1$.*
3. *$[\mathcal{Q}]$ is a Q-neighbor of $[\mathcal{P}]$ if there are $\mathcal{P}' \in [\mathcal{P}]$ and $\mathcal{Q}' \in [\mathcal{Q}]$ such that $\mathcal{Q}'$ is a Q-neighbor of $\mathcal{P}'$, or $\mathcal{P}'$ is a $Q^\ominus$-neighbor of $\mathcal{Q}'$;*

4. $[\mathcal{Q}]$ *is a Q-reachable path of* $[\mathcal{P}]$ *if there are* $[\mathcal{P}_1], \cdots, [\mathcal{P}_n] \in \mathscr{P}$ *such that* $[\mathcal{P}_i] \neq [\mathcal{P}_j]$ *for* $1 \leq i < j \leq n$, $[\mathcal{P}] = [\mathcal{P}_1]$, $[\mathcal{Q}] = [\mathcal{P}_n]$, $[\mathcal{P}_2]$ *is the* ρ*-neighbor of* $[\mathcal{P}_1]$*, and* $[\mathcal{P}_{i+1}]$ *is a Q-neighbor of* $[\mathcal{P}_i]$ *for all* $1 \leq i < n$. $\triangleleft$

Since two paths $\mathcal{P}$ and $\mathcal{P}'$ meet at the same star-type (i.e. $\mathsf{tail}_\sigma(\mathcal{P}) = \mathsf{tail}_\sigma(\mathcal{P}')$) and the same layer (i.e. $\mathsf{tail}_\delta(\mathcal{P}) = \mathsf{tail}_\delta(\mathcal{P}')$) should be considered as identical, we define the equivalence relation $\sim$ in Definition 7 to formalize this idea. Note that for two paths $\mathcal{P}, \mathcal{P}'$ with $\mathsf{tail}_\rho(\mathcal{P}) \neq \mathsf{tail}_\rho(\mathcal{P}')$, we have $\mathcal{P} \sim \mathcal{P}'$ if $\mathsf{tail}_\sigma(\mathcal{P}) = \mathsf{tail}_\sigma(\mathcal{P}')$ and $\mathsf{tail}_\delta(\mathcal{P}) = \mathsf{tail}_\delta(\mathcal{P}')$. This does not allow for extending $\mathsf{tail}_\rho(\mathcal{P})$ to $\mathsf{tail}_\rho([\mathcal{P}])$. As a consequence, there may be several "predecessors" of an equivalence class $[\mathcal{P}]$. However, we can define $\mathsf{tail}_\sigma([\mathcal{P}]) = \mathsf{tail}_\sigma(\mathcal{P})$, $\mathsf{tail}_\delta([\mathcal{P}]) = \mathsf{tail}_\delta(\mathcal{P})$ and $\mathcal{L}([\mathcal{P}]) = \mathcal{L}(\mathcal{P})$. In the sequel, we use $\mathcal{P}$ instead of $[\mathcal{P}]$ whenever it is clear from the context.

In a tree-shaped structure where each node has a unique predecessor, each path $\mathcal{P}$ is identical to its equivalence class $[\mathcal{P}]$. This no longer holds for the general graph structure. The notion of paths in a frame is needed to define cycles which are crucial to establish termination condition when building a frame.

Definition 8 (cycle). *Let* $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ *be a frame for a* $\mathcal{SHOIQ}_{(+)}$ *knowledge base* $(\mathcal{T}, \mathcal{R})$ *with a set* $\mathscr{P}$ *of paths in* $\mathcal{F}$. *Let* $\mathscr{R}$ *be a set of pairs* $(\mathcal{P}_r, \xi_r)$*, called* root paths*, where* $\mathcal{P}_r \in \mathscr{P}$ *and* $\xi_r \subseteq \xi(\mathsf{tail}_\sigma(\mathcal{P}_r))$. *Let* Θ *be a set of quadruples* $(\mathcal{P}, \rho, \mathcal{Q}, \nu)$ *where* $\mathcal{P}, \mathcal{Q} \in \mathscr{P}$ $(\mathcal{P} \neq \mathcal{Q})$*, respectively called* cycled *and* cycling *paths of* Θ *,* $\rho \in \xi(\mathsf{tail}_\sigma(\mathcal{P}))$*,* $\nu \in \xi(\mathsf{tail}_\sigma(\mathcal{Q}))$*, respectively called* cycled *and* cycling *rays of* Θ*. A* ρ*-neighbor of a cycled (*resp. *cycling) path* $\mathcal{P}$ *is a cycled (*resp. *cycling) neighbor of* $\mathcal{P}$ *if* ρ *is a cycled (*resp. *cycling) ray of* $\mathcal{P}$*. We say that* Θ *is a* cycle *w.r.t. a set* $\mathscr{R}$ *of root paths if for each quadruple* $(\mathcal{P}, \rho, \mathcal{Q}, \nu) \in \Theta$ *the following conditions are satisfied:*

1. $o \notin \mathcal{L}(\mathcal{P}) \cup \mathcal{L}(\mathcal{Q}) \cup \bigcup_{\rho \in \xi(\mathsf{tail}_\sigma(\mathcal{P})) \cup \xi(\mathsf{tail}_\sigma(\mathcal{Q}))} l(\rho)$ *for all* $o \in \mathbf{C}_o$*;*
2. $\mathcal{L}(\mathcal{P}) = l(\nu)$, $\mathcal{L}(\mathcal{Q}) = l(\rho)$ *and* $r(\rho) = r^-(\nu)$.
3. *for each ray* $\rho' \in \xi(\mathsf{tail}_\sigma(\mathcal{P}))$ *that is not cycled, there are a sequence* $\mathcal{P}_1, \cdots, \mathcal{P}_n \in \mathscr{P}$*, some* $(\mathcal{P}_0, \rho_0, \mathcal{Q}_0, \nu_0) \in \Theta$ *and a root path* $(\mathcal{P}_r, \xi_r) \in \mathscr{R}$ *such that* $\mathcal{P}_i \neq \mathcal{P}_j$ *for* $1 \leq i < j \leq n$, $\mathcal{P}_1 = \mathcal{P}$, $\mathcal{P}_2$ *is the* ρ'*-neighbor of* $\mathcal{P}_1$*,* $\mathcal{P}_{i+1}$ *is a neighbor of* $\mathcal{P}_i$ *for* $1 \leq i < n$, $\mathcal{P}_k = \mathcal{Q}_0$ *for some* $1 < k < n-1$*, and* $\mathcal{P}_n = \mathcal{P}_r$, $\mathcal{P}_{n-1}$ *is a* ρ_r*-neighbor of* $\mathcal{P}_n$ *with* $\rho_r \in \xi_r$.
4. *for each ray* $\nu' \in \xi(\mathsf{tail}_\sigma(\mathcal{Q}))$ *that is not cycling and each sequence* $\mathcal{P}_1, \cdots, \mathcal{P}_n \in \mathscr{P}$ *such that* $\mathcal{P}_i \neq \mathcal{P}_j$ *for* $1 \leq i < j \leq n$, $\mathcal{P}_1 = \mathcal{Q}$, $\mathcal{P}_2$ *is the* ν'*-neighbor of* $\mathcal{Q}$*, and* $\mathcal{P}_{i+1}$ *is a neighbor of* $\mathcal{P}_i$ *for* $1 \leq i < n$*, there is some* $(\mathcal{P}_0, \rho_0, \mathcal{Q}_0, \nu_0) \in \Theta$ *such that one of the following conditions is satisfied:*
   (a) *there is some* $1 < k \leq n$ *with* $\mathcal{P}_k = \mathcal{Q}_0$ *or* $\mathcal{P}_k = \mathcal{P}_0$*, and* $\mathcal{P}_i$ *is not a cycling and cycled neighbor for all* $1 \leq i \leq k$*;*
   (b) *there are* $\mathcal{P}_{n+1}, \cdots, \mathcal{P}_{n+m} \in \mathscr{P}$ *with* $\mathcal{P}_0 = \mathcal{P}_{n+m}$ *or* $\mathcal{Q}_0 = \mathcal{P}_{n+m}$ *such that* $\mathcal{P}_i \neq \mathcal{P}_j$ *for* $1 \leq i < j \leq n+m$, $\mathcal{P}_{i+1}$ *is a neighbor of* $\mathcal{P}_i$ *for all* $n \leq i < n+m$*, and* $\mathcal{P}_i$ *is not a cycling and cycled neighbor for all* $1 \leq i \leq n+m$*;*

We use $\mathscr{R}_0$ *to denote the set of all pairs* $(\mathcal{P}_r, \xi(\mathsf{tail}_\sigma(\mathcal{P}_r)))$ *where* $\mathcal{P}_r$ *is a nominal path. A* primary cycle Θ_0 *is a cycle w.r.t.* $\mathscr{R}_0$*. Furthermore, we define a* reachable cycle Θ' *of a cycle of* Θ *if* Θ' *is a cycle w.r.t. the set of all pairs* $(\mathcal{P}_r, \xi_r)$ *where* $\mathcal{P}_r$ *is a cycled path of* Θ *and* ξ_r *is the set of all cycled rays of* $\mathcal{P}_r$*.*

Note that a cycle Θ may encapsulate a *loop* if it includes two quadruples $(\mathcal{P}, \rho, \mathcal{Q}, \nu)$, $(\mathcal{P}', \rho', \mathcal{Q}', \nu')$ such that $\mathcal{Q}'$ is a reachable path of $\mathcal{Q}$ via ρ. A loop can be formed from a sequence $\mathcal{P}_1, \cdots, \mathcal{P}_n \in \mathscr{P}$ $(n > 3)$ such that $\mathcal{P}_1 = \mathcal{P}_n$, $\mathcal{P}_i \neq \mathcal{P}_j$ for $1 \leq i < j < n$ and $\mathcal{P}_{i+1}$ is a neighbor of $\mathcal{P}_i$ for $1 \leq i < n$). Moreover, it is possible that there are two quadruples $(\mathcal{P}, \rho, \mathcal{Q}, \nu), (\mathcal{P}', \rho', \mathcal{Q}', \nu') \in \Theta$ such that $\mathcal{Q}' = \mathcal{Q}$, $\nu = \nu'$ and $\mathcal{P}' \neq \mathcal{P}$, $\rho \neq \rho'$, or $\mathcal{P}' = \mathcal{P}$, $\rho = \rho'$ and $\mathcal{Q}' \neq \mathcal{Q}$, $\nu \neq \nu'$.

Intuitively, a (primary) cycle allows one to "cut" all paths started from nominal paths of a frame into two parts : the first path which is connected to nominal paths is not replicated while the second part can be infinitely lengthened. Condition 1, Definition 8 says that a cycle should not include nominal star-types which must not replicated. Condition 2 says that a cycled path "matches" its cycling path via a ray with the same label. Condition 3 not only provides the relationship between two paths $\mathcal{P}, \mathcal{Q}$ for each $(\mathcal{P}, \rho, \mathcal{Q}, \nu) \in \Theta$ but also ensures that all *non-cycled* neighbors of each $\mathcal{P}$ are filled in a cycle. Condition 4 ensures that an extension of cycled paths $\mathcal{P}$ via their cycled neighbors is possible by replicating paths from its cycling path $\mathcal{Q}$ via cycling rays.

As a consequence, the existence of a cycle allows one to "unravel" a set $\mathscr{P}$ of paths in a frame to obtain a possibly infinite set $\widehat{\mathscr{P}}$ of paths. The following lemma characterizes this crucial property and provides a bound on the size of a cycle.

Lemma 3. *Let $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ be a frame for a $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$. Let Θ be a cycle in $\mathcal{F}$.*

1. *There exists an extension $\widehat{\mathscr{P}_\Theta}$ of paths between cycled and cycling paths such that each path $\mathcal{P}_0 \in \widehat{\mathscr{P}_\Theta}$ has exactly $|\xi(\mathsf{tail}_\sigma(\mathcal{P}_0))|$ neighbors.*
2. *If Θ' is a reachable cycle of Θ then $|\Theta'| \leq |\Theta| \times |\xi|^{2^\ell}$ where $|\xi|$ is the maximal number of rays of a star-type, and $\ell = 2^{2 \times |\mathbf{CL}(\mathcal{T}, \mathcal{R})| \times |\mathbf{R}(\mathcal{T}, \mathcal{R})|}$.*

A proof of Lemma 3 can be based on the fact that all paths between cycling and cycled paths of a cycle do not cross the borders defined by the cycle. Therefore, these paths can be replicated and pasted to cycled paths. With regard to the size of a cycle, we can use the following construction: each path starts from a nominal star-type in $\mathcal{N}_o$ and is lengthened through star-types (more precisely, through layers of rays of star-types). We define inductively a level n of a path $\mathcal{P}$ as follows: (i) all nominal paths are at level 0, (ii) a path $\mathcal{P}'$ is at level $i+1$ if it has a neighbor at level i, and all neighbors of $\mathcal{P}'$ are at a level which are equal or greater than i. This implies that there are no two neighbor paths which are located on two levels whose difference is greater than 1.

Assume that there is a pair of paths $(\mathcal{Q}, \mathcal{Q}')$ such that $\mathcal{Q}$ is at level $i > 1$ and $\mathcal{Q}'$ is a ν-neighbor of $\mathcal{Q}$ at level $i-1$ iff there is a pair of paths $(\mathcal{P}, \mathcal{P}')$ such that $\mathcal{P}$ is at level $j > i$, $\mathcal{P}'$ is a ρ-neighbor of $\mathcal{P}$ at level $j+1$, and $\mathcal{L}(\mathcal{Q}) = \mathcal{L}(\mathcal{P}')$, $\mathcal{L}(\mathcal{Q}') = \mathcal{L}(\mathcal{P})$, $r(\nu) = r^-(\rho)$. This implies that all such quadruples $(\mathcal{P}, \rho, \mathcal{Q}, \nu)$ can form a cycle. Moreover, there are at most ℓ different labels of pairs $(\mathcal{Q}, \nu)$. This implies that one cycle can be detected after creating at most 2^ℓ levels. Thus, we have $|\Theta'| \leq |\Theta| \times |\xi|^{2^\ell}$ where $|\xi|$ is the maximal number of rays of star-type. A more complete proof of Lemma 3 can be found in [14].

Let Θ be a cycle in a frame. Definition 8 ensures that each reachable path of some path $\mathcal{Q}$ with $(\mathcal{P}, \rho, \mathcal{Q}, \nu) \in \Theta$ goes through a star-type $\sigma = \mathsf{tail}_\sigma(\mathcal{P}')$ with some

$(\mathcal{P}', \rho', \mathcal{Q}', \nu') \in \Theta$. As mentioned in Lemma 3, such a cycle allows one to "unravel" infinitely the frame to obtain a model of a KB in $\mathcal{SHOIQ}$ (without transitive closure of roles). However, such a cycle structure is not sufficient to represent models of a KB with transitive closure of roles since a concept such as $\exists Q^{\oplus}.D \in \mathcal{L}(\mathcal{P})$ can be satisfied by a Q-reachable path $\mathcal{P}'$ of $\mathcal{P}$ which is arbitrarily far from $\mathcal{P}$. There are the following possibilities for an algorithm which builds a frame: (i) the algorithm stops building the frame as soon as a cycle Θ is detected such that each concept of the form $\exists Q^{\oplus}.D$ occurring in $\mathcal{L}(\mathcal{P})$ is satisfied for each cycled path $\mathcal{P}$ of Θ, i.e., $\mathcal{P}$ has a Q-reachable path $\mathcal{P}'$ with $\exists Q.D \in \mathcal{L}(\mathcal{P})$, (ii) despite of several detected cycles, the algorithm continues building the frame until each concept of the form $\exists Q^{\oplus}.D$ occurring in $\mathcal{L}(\mathcal{P})$ is satisfied for each cycled path $\mathcal{P}$ of Θ. If we adopt the first possibility, the completeness of such an algorithm cannot be established since there are models in which paths satisfying concepts of the form $\exists Q^{\oplus}.D$ can spread over several "iterative structures" such as cycles. For this reason, we adopt the second possibility by introducing into frames an additional structure, namely *blocking-blocked cycles*, which determines a sequence of cycles $\Theta_1, \cdots, \Theta_k$ such that Θ_{i+1} is a reachable cycle of Θ_i for satisfying concepts of the form $\exists Q^{\oplus}.D$.

Definition 9 (blocking). *Let $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ be a frame for a $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$ with a set $\mathscr{P}$ of paths in $\mathcal{F}$. A cycle Θ' is* blocked *by a cycle Θ if there are cycles $\Theta_1, \cdots, \Theta_k$ with $\Theta = \Theta_1$, $\Theta' = \Theta_k$ such that Θ_{i+1} is a reachable cycle of Θ_i for $1 \leq i < k$, and the following conditions are satisfied:*

1. *For each $1 \leq i < k$, there is no cycle Θ'' such that*
   (a) *Θ'' is a reachable cycle of Θ_i and Θ_{i+1} is a reachable cycle of Θ'', and*
   (b) *For each $(\mathcal{P}, \rho, \mathcal{Q}, \nu) \in \Theta''$ and each concept $\exists Q^{\oplus}.D \in \mathcal{L}(\mathcal{P})$, $\mathcal{P}$ has a Q-reachable path $\mathcal{P}'$ via a non cycled ray with $\exists Q.D \in \mathcal{L}(\mathcal{P}')$ iff the ν-neighbor $\mathcal{Q}'$ of $\mathcal{Q}$ has a Q-reachable path $\mathcal{Q}''$ via a non cycling ray with $\exists Q.D \in \mathcal{L}(\mathcal{Q}'')$.*
2. *For each $(\mathcal{P}_k, \rho_k, \mathcal{Q}_k, \nu_k) \in \Theta_k$, there is some $(\mathcal{P}_1, \rho_1, \mathcal{Q}_1, \nu_1) \in \Theta_1$ such that*
   (a) *$\mathcal{L}(\mathcal{P}_1) = \mathcal{L}(\mathcal{P}_k)$, $\mathcal{L}(\mathcal{Q}_1) = \mathcal{L}(\mathcal{Q}_k)$, $r(\rho_1) = r(\rho_k)$, and*
   (b) *If there is a concept $\exists Q^{\oplus}.D \in \mathcal{L}(\mathcal{P}_k)$ such that the path $\mathcal{P}_k$ has no Q-reachable path $\mathcal{P}'$ with $\exists Q.D \in \mathcal{L}(\mathcal{Q}')$ then the path $\mathcal{Q}_1$ has a Q-reachable path $\mathcal{Q}$ such that the two following conditions are satisfied:*
      i. *$\exists Q.D \in \mathcal{L}(\mathcal{Q})$, or $\mathcal{Q}$ has a Q-reachable path $\mathcal{Q}'$ with $\exists Q.D \in \mathcal{L}(\mathcal{Q}')$,*
      ii. *there are $(\mathcal{P}_j, \rho_j, \mathcal{Q}_j, \nu_j) \in \Theta_j$, $(\mathcal{P}_{j+1}, \rho_{j+1}, \mathcal{Q}_{j+1}, \nu_{j+1}) \in \Theta_{j+1}$ with some $1 \leq j < k$ such that $\mathcal{Q}'$ is a reachable path of $\mathcal{Q}_j$ and $\mathcal{Q}_{j+1}$ is a reachable path of $\mathcal{Q}'$.*

   In this case, we say that the path $\mathcal{P}_k$ is blocked by the path $\mathcal{Q}_1$ via the ray ρ_k. $\triangleleft$

Definition 9 provides an exact structure of a frame in which blocked paths can be detected. Such a frame contains sequentially reachable cycles between a blocking cycle Θ_1 and its blocked cycle Θ_k, which allows for unravelling the frame between Θ_k and Θ_1, and satisfying all concepts of the form $\exists Q^{\oplus}.D$ in the labels of paths in Θ_1. Condition 1 ensures that there is no useless cycle for the satisfaction of concepts $\exists Q^{\oplus}.D$ which is located between two cycles Θ_i and Θ_i with $i < k$. For a concept $\exists Q^{\oplus}.D \in \mathcal{L}(\mathcal{P}_k)$ that is not satisfied from the path $\mathcal{P}_k$ to all existing paths (i.e. it is not satisfied in the "past"), it must be satisfied from $\mathcal{P}_k$ to paths that are devised by unravelling (i.e. it is satisfied in "the future"). Therefore, it is required that such concepts $\exists Q^{\oplus}.D$ are satisfied in the "future" from the blocking path $\mathcal{P}_1$ of $\mathcal{P}_k$ (Condition 2, Definition 9). Moreover, for a concept $\exists Q^{\oplus}.D \in \mathcal{L}(\mathcal{P})$ that is not satisfied in the "past", either it is satisfied from $\mathcal{P}$ to some paths that are explicitly added to the frame, or it is propagated to a some blocked path thanks to Property 11, Definition 4.

Remark 2. The constant k mentioned in Definition 9 depends to the number of distinct ray labels (i.e. the triple $\langle \mathcal{L}(\mathcal{P}), r(\rho), l(\rho)\rangle$ for each ray $\rho \in \mathsf{tail}_\sigma(\mathcal{P})$) occurring a blocking cycle Θ_1 and the number of concepts $\exists Q^{\oplus}.D$ occurring in each cycling path label in Θ_1. Since the number of distinct ray labels is bounded by ℓ (Lemma 3) and the number of concepts $\exists Q^{\oplus}.D$ occurring in each cycling path label is bounded by $\mathbf{CL}(\mathcal{T},\mathcal{R})$, we have k is bounded by $2 \times \ell$ where $\ell = 2^{2\times|\mathbf{CL}(\mathcal{T},\mathcal{R})|\times|\mathbf{R}(\mathcal{T},\mathcal{R})|}$.

Definition 10 (valid frame). *Let $(\mathcal{T},\mathcal{R})$ be a $\mathcal{SHOIQ}$ knowledge base. A frame $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta\rangle$ with a set $\mathscr{P}$ of paths is* valid *if the following conditions are satisfied:*

1. *For each nominal $o \in \mathbf{C}_o$, there is a unique $\sigma_o \in \mathcal{N}_o$ such that $o \in \lambda(\sigma_o)$ and $\delta(\sigma_o) = 1$;*
2. *For each star-type $\sigma \in \mathcal{N}$, σ is valid.*
3. *If $\exists Q^{\oplus}.C \in \mathcal{L}(\mathcal{P}_0)$ for some $\mathcal{P}_0 \in \mathscr{P}$ then there are $\mathcal{P}, \mathcal{P}' \in \mathscr{P}$ such that one of the following conditions is satisfied:*
   (a) $\mathcal{P}_0 = \mathcal{P} = \mathcal{P}'$ and $\exists Q.C \in \mathcal{L}(\mathcal{P}_0)$;
   (b) $\mathcal{P}'$ is a Q-reachable of $\mathcal{P}$, and $\exists Q.C \in \mathcal{L}(\mathcal{P}')$ where $\mathcal{P} = \mathcal{P}_0$ or $\mathcal{P}$ blocks $\mathcal{P}_0$;
   (c) $\mathcal{P}$ is a $Q^{\ominus}$-reachable of $\mathcal{P}'$, and $\exists Q.C \in \mathcal{L}(\mathcal{P}')$ where $\mathcal{P} = \mathcal{P}_0$ or $\mathcal{P}$ blocks $\mathcal{P}_0$. $\triangleleft$

Conditions 1-3 in Definition 10 ensure the satisfaction of tableau properties in Definition 3. Note that Condition 1 is compatible with the fact that cycles in a frame never consist of nominal star-types (Definition 8). In particular, Condition 3 provides the satisfaction of concepts $\exists Q^{\oplus}.D$ occurring in the labels of paths thanks to the blocking condition introduced in Definition 9.

We now present Algorithm 1 for building a valid frame. This algorithm starts by adding nominal star-types to the frame. For each non blocked path $\mathcal{P}$ with a ray $\rho \in \xi(\mathsf{tail}_\sigma(\mathcal{P}))$ such that $\delta(\mathsf{tail}_\sigma(\mathcal{P}), \rho) = \delta(\mathsf{tail}_\sigma(\mathcal{P})) + 1$, the algorithm picks in a nondeterministic way a valid star-type ω that matches $\mathsf{tail}_\sigma(\mathcal{P})$ via ρ, and updates the values $\Omega(\mathsf{tail}_\sigma(\mathcal{P}), \rho)$, $\Omega(\omega, \rho')$, $\delta(\mathsf{tail}_\sigma(\mathcal{P}), \rho)$, $\delta(\omega, \rho')$, eventually, $\delta(\mathsf{tail}_\sigma(\mathcal{P}))$ and $\delta(\omega)$ by calling $\mathsf{updateFrame}(\cdots)$. The algorithm terminates when a blocked cycle is detected. To check the blocking condition, the algorithm can compare $\mathscr{R}_i$ for each new level i of rays with each $\mathscr{R}_j$ for all $j < i$ (the notion of levels of rays in a frame is given in the proof of Lemma 3) where $\mathscr{R}_j$ is denoted for the set of different ray labels at level i. If $\mathscr{R}_j = \mathscr{R}_i$ and the last cycle that was detected located at some level $l < j$, then a new (reachable) cycle from level j to i is formed.

Figure 2 depicts a frame when executing Algorithm 1 for $\mathcal{K}_1$ in the example presented in Section 1. The algorithm builds a frame $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta\rangle$ where $\mathcal{N} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ and $\mathcal{N}_o = \{\sigma_0\}$. The dashed arrows indicate how the function $\Omega(\sigma, \rho)$ can be built. For example, $\Omega(\sigma_0, \rho_0) = \{(\sigma_1, \nu_0, 1)\}$, $\Omega(\sigma_0, \rho_1) = \{(\sigma_2, \rho'_0, 1)\}$

Require: A $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$
Ensure: A frame $\langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ for $(\mathcal{T}, \mathcal{R})$
Let Σ be the set of all star-types for $(\mathcal{T}, \mathcal{R})$
for all $o \in \mathbf{C}_o$ **do**
    if there is no $\sigma \in \mathcal{N}$ such that $o \in \lambda(\sigma)$ **then**
        Choose a star-type $\sigma_o \in \Sigma$ such that $o \in \lambda(\sigma_o)$
        Set $\delta(\sigma_o) = 1$, $\mathcal{N} = \mathcal{N} \cup \{\sigma_o\}$ and $\mathcal{N}_o = \mathcal{N}_o \cup \{\sigma_o\}$
        Set $\delta(\sigma_o, \rho) = 0$, $\Omega(\sigma_o, \rho) = \emptyset$ for all $\rho \in \xi(\sigma_o)$
    end if
end for
while there is a path $\mathcal{P}$ that is not blocked and a ray $\rho \in \xi(\mathsf{tail}_\sigma(\mathcal{P}))$ such that $\mathsf{tail}_\delta(\mathcal{P}) = \delta(\mathsf{tail}_\sigma(\mathcal{P}), \rho) + 1$ **do**
    Choose a star-type $\sigma' \in \Sigma$ such that there is a ray $\rho' \in \xi(\sigma')$ satisfying
        $l(\rho) = \lambda(\sigma')$, $l(\rho') = \lambda(\sigma)$, $r(\rho') = r^-(\rho)$, and
        $\sigma' \in \mathcal{N}$ implies $\delta(\sigma') = \delta(\sigma', \rho') + 1$ or $\delta(\sigma') = \delta(\sigma', \rho'')$ for all $\rho'' \in \xi(\sigma')$
    $\mathsf{updateFrame}(\sigma, \rho, \sigma', \rho')$
end while

Algorithm 1: An algorithm for building a frame

Require: A star-type $\sigma \in \mathcal{N}$ in a frame $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ with a ray $\rho \in \xi(\sigma)$, and a new star-type σ' with a ray $\rho' \in \xi(\sigma')$ such that $l(\rho) = \lambda(\sigma')$, $l(\rho') = \lambda(\sigma)$, $r(\rho') = r^-(\rho)$
Ensure: $\mathsf{updateFrame}(\sigma, \rho, \sigma', \rho')$
if there exists a star-type $\omega \in \mathcal{N}$ such that ω is equivalent to σ' **then**
    Set $\delta(\sigma, \rho) = \delta(\sigma, \rho) + 1$
    Let $\nu \in \xi(\omega)$ such that $r(\nu) = r(\rho')$ and $l(\nu) = l(\rho')$
    if $\delta(\omega, \nu) == \delta(\omega)$ **then**
        Set $\delta(\omega) = \delta(\omega) + 1$
    end if
    Set $\delta(\omega, \nu) = \delta(\omega, \nu) + 1$
    $\mathsf{add}(\Omega(\omega, \nu), (\sigma, \rho, \delta(\sigma, \rho)))$
    $\mathsf{add}(\Omega(\sigma, \rho), (\omega, \nu, \delta(\omega, \nu)))$
else
    Add σ' to $\mathcal{N}$
    Set $\delta(\sigma, \rho)$=$\delta(\sigma, \rho) + 1$
    Set $\delta(\sigma') = 1$, $\delta(\sigma', \rho') = 1$ and $\Omega(\sigma', \rho') = \{(\sigma, \rho, \delta(\sigma, \rho))\}$
    Set $\delta(\sigma', \rho'') = 0$ and $\Omega(\sigma', \rho'') = \emptyset$ for all $\rho'' \neq \rho'$
    $\mathsf{add}(\Omega(\sigma, \rho), (\sigma', \rho', 1))$
end if

Algorithm 2: $\mathsf{updateFrame}(\sigma, \rho, \sigma', \rho')$ updates $\mathcal{F}$ when adding σ' to $\mathcal{N}$

where ρ_0 and ρ_1 are the respective horizontal and vertical rays of σ_0; ν_0 is the left ray of σ_1; ρ'_0 is the vertical ray of σ_2. Moreover, the directed dashed arrow from σ_0 to σ_1 indicates that the ray ρ_0 of σ_0 can match the ray ν_0 on the left ray of σ_1 since $l(\rho_0) = \lambda(\sigma_1)$, $r(\nu_0) = \lambda(\sigma_0)$, $r(\nu_0) = r^-(\rho_0)$.

The algorithm generates $\delta(\sigma_0) = 1$, $\delta(\sigma_1) = 1$, $\delta(\sigma_2) = 1$ and forms a cycle Θ consisting of the following quadruples : $((\sigma_3, 3), \rho_1, (\sigma_3, 2), \rho_2)$ (ρ_1 and ρ_2 are the right and left rays of σ_3, respectively) and $((\sigma_4, 2), \rho_3, (\sigma_4, 1), \rho_4)$ (ρ_3 and ρ_4 are the right and left rays of σ_4 respectively). Note that for the sake of brevity, we use just $\mathsf{tail}_\sigma(\mathcal{P})$ and $\mathsf{tail}_\delta(\mathcal{P})$ to denote a path in the quadruples.

The cycle Θ is blocked since all concepts $\exists S^+.\{o\}$ occurring in cycled paths are satisfied. A model of the ontology can be built by starting from σ_0 and getting (i) σ_4 via σ_1, (ii) σ_3 via σ_1, and (iii) σ_3 via σ_2. From σ_3 and σ_4, the model goes through σ_3 and σ_4 infinitely. Note that from any individual x satisfying σ_3 (or σ_4), i.e. the "label" of x contains $\exists Q^+.\{o\}$, there is a path containing S which goes back the individual satisfying σ_0. Thus, the concept $\exists Q^+.\{o\}$ is satisfied for each individual whose label contains $\exists Q^+.\{o\}$.

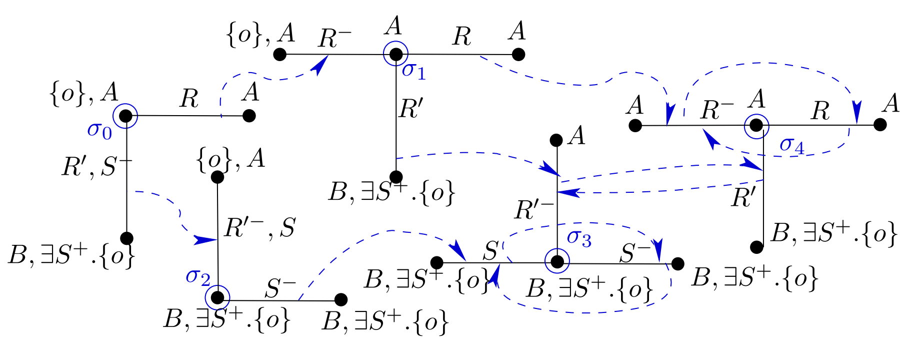

Fig. 2. A frame obtained by Algorithm 1 for $\mathcal{K}_1$ in the example in Section 1

Lemma 4. *Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}_{(+)}$ knowledge base.*

1. *Algorithm 1 terminates.*
2. *If Algorithm 1 can build a valid frame for $(\mathcal{T}, \mathcal{R})$ then there is a tableau for $(\mathcal{T}, \mathcal{R})$.*
3. *If there is a tableau for $(\mathcal{T}, \mathcal{R})$ then Algorithm 1 can build a valid frame $\mathcal{F}$ for $(\mathcal{T}, \mathcal{R})$.*

Proof (sketch). Let Θ_k be a blocked cycle by Θ_1. According to Remark 2, k is bounded by $\mathcal{O}(2^{|(\mathcal{T},\mathcal{R})|})$. Moreover, after eliminating "useless cycles" between two cycles Θ_i and Θ_{i+1} for $1 \leq i < k$ according to Condition 1, Definition 9 the number of useful cycles between Θ_i and Θ_{i+1} is bounded by $\mathcal{O}(2^{2^{|(\mathcal{T},\mathcal{R})|}})$. This implies that Algorithm 1 can add at most a triple exponential number of paths to the frame to form a blocked cycle. For the soundness of Algorithm 1, we can extend the set $\mathscr{P}$ of paths to a set $\widehat{\mathscr{P}}$

of extended paths by "unravelling" the frame between blocking-blocked cycles. The set $\widehat{\mathscr{P}}$ allows one to satisfy concepts $\exists Q^{\oplus}.D$ in blocked paths which are not satisfied in the "past". Moreover, a concept $\exists Q^{\oplus}.D$ of a path that is not satisfied in the "past" will be propagated to a blocked path via a Q-path. Therefore, it will be satisfied in $\widehat{\mathscr{P}}$. Unlike the unravelling of a completion graph for $\mathcal{SHOIQ}$ where there is no loop in the model, the unravelling of a frame may yield an infinite number of loops in the model. Note that the unravelling of a frame replicates cycles which may encapsulate loops.

Regarding completeness, we first reduce a tableau to a frame that does not contain any useless cycle. Then, we use the obtained frame to guide the algorithm (i) to choose valid star-types, (ii) to ensure that $\delta(\sigma) = 1$ for each nominal star-type σ, and (iii) to detect a pair (Θ_1, Θ_k) of blocking and blocked cycles as soon as some "representative" concepts of the form $\exists Q^{\oplus}.D$ in Θ_1 are satisfied. We refer the readers to [14] for a complete proof of Lemma 4. □

The following theorem is a consequence of Lemma 4.

Theorem 1. *The problem of consistency for $\mathcal{SHOIQ}_{(+)}$ can be decided in non-deterministic triply exponential time in the size of a $\mathcal{SHOIQ}_{(+)}$ knowledge base.*

4 Optimizing The Algorithm

The algorithm for deciding the consistency of a $\mathcal{SHOIQ}_{(+)}$ knowledge base (Algorithm 1) uses at most a doubly exponential number of star-types to build a frame. This is due to the fact that numbers are encoded in binary, that is, a star-type may have an exponential number of rays. Pratt-Hartmann [9] has shown that it is possible to use an exponential number of star-types to represent a model of a KB in $\mathcal{C}^2$ which is slightly different from $\mathcal{SHOIQ}$ in terms of expressiveness. If we can transfer this method to $\mathcal{SHOIQ}$ for compressing star-types, it would be applied to $\mathcal{SHOIQ}_{(+)}$ since the number of star-types in a frame does not depend on the presence of transitive closure of roles.

Another technique presented in [15] can be used to reduce non-determinisms due to the choice of valid star-types. Instead of guessing a valid star-type from a set of valid star-types, this technique allows one to build a star-type σ by applying expansion rules to concepts in the core label of σ. Hence, when a star-type σ is transformed into σ' by an expansion rule, an algorithm that implements this technique has to update not only $\Omega(\sigma', \rho')$ and $\delta(\sigma')$ but also $\Omega(\sigma'', \rho'')$ and $\delta(\sigma'')$ for each neighbor σ'' of σ and σ' (σ'' is a neighbor of σ' if there is some $(\sigma'', \rho'', d'') \in \Omega(\sigma', \rho')$). These updates must ensure that each path which has got through σ can now get through σ'. This process of changes can spread over neighbors of σ'' and so on.

With regard to blocking, the technique presented in [15] can take advantage of a specific structure of frames for $\mathcal{SHOIQ}$ to design an efficient algorithm for checking blocking condition. This structure consists of partitioning star-types into layers. Although such a structure of frames cannot be maintained for $\mathcal{SHOIQ}_{(+)}$, paths in a frame for $\mathcal{SHOIQ}_{(+)}$ would allow us to achieve the same behavior.

5 Conclusion

In this paper, we have presented a decision procedure for the description logic $\mathcal{SHOIQ}$ with transitive closure of roles in concept axioms, whose decidability was not known. The most significant feature of our contribution is to introduce a structure based on a new blocking condition for characterizing models which have an infinite non-tree-shaped part. This structure would provide an insight into regularity of such models which would be enjoyed by a more expressive DL, such as $\mathcal{ZOIQ}$ [6], whose decidability remains open. In future work, we aim to improve the algorithm by making it more goal-directed and aim to investigate another open question about the hardness of $\mathcal{SHOIQ}_{(+)}$.

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